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Strong Approximation by Dirichlet Integrals in $L^\lambda(\mathbf{R})$-Norm, $1 < \lambda < \infty$

DANG VU GIANG*

AND

FERENC MÓRICZ*

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

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Let f be a real valued function which belongs to $L^r := L^r(-\infty, \infty)$ for some $1 \le r < \infty$. We consider the cosine transform $\hat{f}_c$, sine transform $\hat{f}_s$, (complex) Fourier transform $\hat{f}$, and Hilbert transform $\tilde{f}$ of f. We study the strong approximation of order p, $0 < p < \infty$, of f and $\tilde{f}$ by their Dirichlet integrals, respectively. We prove that the saturation class in L^λ-norm is the Lizorkin–Triebel space $F^\alpha_{\lambda, p}$, where $\alpha = 1/p$, $2 \le p < \infty$, and $1 < \lambda < \infty$. To this effect, we introduce several so-called Littlewood–Paley functions and make use of a number of equivalence theorems. Our machinery is also appropriate to characterize the saturation class concerning the strong approximation of order p of a periodic function $f \in L^1_{2\pi} := L^1(-\pi, \pi)$ by the partial sums of its Fourier series in $L^\lambda_{2\pi}$-norm, where again $2 \le p < \infty$ and $1 < \lambda < \infty$.

1. PRELIMINARIES

We recall the basic notions in the theory of Fourier transforms (see, e.g., [6, Chaps. 1–5]). Let f be a real valued, Lebesgue integrable function on the real line, in sign: $f \in L^1 := L^1(\mathbf{R})$, $\mathbf{R} := (-\infty, \infty)$. Let

$$\hat{f}_c(u) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ut \, dt \tag{1.1}$$

be the cosine transform and

$$\hat{f}_s(u) := \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin ut \, dt, \qquad u \in \mathbf{R}, \tag{1.2}$$

the sine transform of f.

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These definitions make sense also in the case when $f \in L^r := L^r(\mathbf{R})$ for some $1 < r \le 2$, since the integrals on the right-hand sides of (1.1) and (1.2) converge in $L^{r'}$-norm, where $1/r + 1/r' = 1$. In other words, the integral $\int_{-\infty}^{\infty}$ is meant as the limit in $L^{r'}$-norm of $\int_{-T_1}^{T_2}$ as $T_1, T_2 \to \infty$. We note that in the case where $f \in L^r$ for some $2 < r < \infty$, then $\hat{f}_c$ and $\hat{f}_s$ exist only in the sense of a tempered distribution, and they are not functions in general.

The Dirichlet integral of a function $f \in L^1$ is defined by

$$s_\nu(f, t) := \int_0^\nu \{\hat{f}_c(u)\cos tu + \hat{f}_s(u)\sin tu\}\, du, \tag{1.3}$$

the conjugate Dirichlet integral by

$$\tilde{s}_\nu(f, t) := \int_0^\nu \{\hat{f}_c(u)\sin tu - \hat{f}_s(u)\cos tu\}\, du, \tag{1.4}$$

the Riesz mean (of first order) by

$$\sigma_\nu(f, t) := \int_0^\nu \left(1 - \frac{u}{\nu}\right)\{\hat{f}_c(u)\cos tu + \hat{f}_s(u)\sin tu\}\, du, \tag{1.5}$$

and the conjugate Riesz mean (of first order) by

$$\tilde{\sigma}_\nu(f, t) := \int_0^\nu \left(1 - \frac{u}{\nu}\right)\{\hat{f}_c(u)\sin tu - \hat{f}_s(u)\cos tu\}\, du, \qquad \nu > 0,\ t \in \mathbf{R}. \tag{1.6}$$

From (1.1)–(1.6) it follows immediately that

$$s_\nu(f, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t-u)\frac{\sin \nu u}{u}\, du, \tag{1.7}$$

$$\tilde{s}_\nu(f, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t-u)\frac{1 - \cos \nu u}{u}\, du, \tag{1.8}$$

which justify the use of the term "Dirichlet integral," and

$$\sigma_\nu(f, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t-u)\frac{1 - \cos \nu u}{\nu u^2}\, du, \tag{1.9}$$

$$\tilde{\sigma}_\nu(f, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t-u)\left(\frac{1}{u} - \frac{\sin \nu u}{\nu u^2}\right) du. \tag{1.10}$$

From now on, it suffices to assume that $f \in L^r$ for some $1 \le r < \infty$, since the integrals on the right-hand sides of (1.7)–(1.10) exist in the

Lebesgue sense, due to Hölder's inequality. Thus, we will use (1.7)–(1.10) in the capacity of the definitions of $s_\nu(f)$, $\tilde{s}_\nu(f)$, $\sigma_\nu(f)$, and $\tilde{\sigma}_\nu(f)$ for functions f belonging to L^r for some $1 \le r < \infty$. Furthermore, we have

$$\sigma_\nu(f,t) = \frac{1}{\nu}\int_0^\nu s_\mu(f,t)\,d\mu, \tag{1.11}$$

$$\tilde{\sigma}_\nu(f,t) = \frac{1}{\nu}\int_0^\nu \tilde{s}_\mu(f,t)\,d\mu, \tag{1.12}$$

which explain that $\sigma_\nu(f)$ and $\tilde{\sigma}_\nu(f)$ are also called the "Cesàro means" of f.

Let $z := x + iy$ be a complex number with $y > 0$. As is known, the Cauchy transform of f is defined by

$$\Phi(f,z) := \int_0^\infty \left\{\hat{f}_c(u) - i\hat{f}_s(u)\right\} e^{izu}\,du. \tag{1.13}$$

It is plain that $\Phi(f,z)$ as a function of the complex variable z is analytic on the upper half plane.

We remind the reader that the Hilbert transform $\tilde{f}$ of a function $f \in L^r$ for some $1 \le r < \infty$ is defined by

$$\tilde{f}(t) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{t-u}\,du = -\lim_{\varepsilon \downarrow 0}\frac{1}{\pi}\int_\varepsilon^\infty \frac{f(t+u) - f(t-u)}{u}\,du. \tag{1.14}$$

This integral as well as the limit

$$\Phi(f,t) := \lim_{y \downarrow 0}\Phi(f, t+iy) = f(t) + i\tilde{f}(t) \tag{1.15}$$

exists and we have

$$(\tilde{f})^{\sim}(t) = -f(t) \tag{1.16}$$

for almost all $t \in \mathbf{R}$. By a famous theorem of M. Riesz, if $f \in L^r$ for some $1 < r < \infty$, then $\tilde{f} \in L^r$ and

$$\tilde{s}_\nu(f,t) = s_\nu(\tilde{f},t), \qquad t \in \mathbf{R}. \tag{1.17}$$

Remark 1. We note that we could equally use complex notations. Namely, the (complex) Fourier transform of a function $f \in L^1$ is defined by

$$\hat{f}(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-iut}\,dt, \qquad u \in \mathbf{R}.$$

By (1.1) and (1.2), we have

$$\hat{f}(u) = \frac{1}{2}\{\hat{f}_c(u) - i\hat{f}_s(u)\}.$$

Assume that $f \in L^r$ for some $1 \le r < \infty$. Then the Dirichlet integrals and Riesz means of f can be expressed as

$$s_\nu(f,t) = \int_{-\nu}^{\nu} \hat{f}(u) e^{iut}\, du,$$

$$\tilde{s}_\nu(f,t) = \int_{-\nu}^{\nu} (-i \operatorname{sign} u) \hat{f}(u) e^{iut}\, du,$$

$$\sigma_\nu(f,t) = \int_{-\nu}^{\nu} \left(1 - \frac{|u|}{\nu}\right) \hat{f}(u) e^{iut}\, du,$$

$$\tilde{\sigma}_\nu(f,t) = \int_{-\nu}^{\nu} \left(1 - \frac{|u|}{\nu}\right)(-i \operatorname{sign} u) \hat{f}(u) e^{iut}\, du, \qquad \nu > 0,\ t \in \mathbf{R},$$

where the left-hand sides are defined by (1.7)–(1.10), respectively.

2. Main Results

Let $f \in L^r$ for some $1 \le r < \infty$ and define

$$\tau_\nu(f,t) := s_\nu(f,t) - \sigma_\nu(f,t),$$
$$\tilde{\tau}_\nu(f,t) := \tilde{s}_\nu(f,t) - \tilde{\sigma}_\nu(f,t), \qquad \nu > 0,\ t \in \mathbf{R}.$$

By (1.11), (1.12), and (1.17), we have

$$\tilde{\tau}_\nu(f,t) = \tau_\nu(\tilde{f},t). \tag{2.1}$$

Furthermore, let $\alpha \in \mathbf{R}$ and define

$$\gamma_*(f,t) := \left\{\int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p \frac{d\nu}{\nu}\right\}^{1/p},$$

$$g_{2*}(f,t) := \left\{\int_0^\infty \left|y^{2-\alpha}\Phi''(f,t+iy)\right|^p \frac{dy}{y}\right\}^{1/p}.$$

THEOREM 1. *Let $f \in L^r$ for some $1 \le r < \infty$, and let $1 < p < \infty$, $-\infty < \alpha < 1$. Then*

$$g_{2*}(f,t) \le C_{\alpha,p}\gamma_*(f,t), \qquad t \in \mathbf{R}. \tag{2.2}$$

Here and in the sequel, by C (endowed with certain subscripts) we denote positive constants (depending only on the subscripts indicated) whose value may be different at different occurrences.

Let $1 < p < \infty$ and define q by $1/p + 1/q = 1$. The main result of this paper reads as follows.

THEOREM 2. *Let $f \in L^p \cap L^\lambda$ for some $1 < p < \infty$, $1 < \lambda < \infty$, and let $0 < \alpha \le \min(1/p, 1/q)$. Then*

$$\gamma_1(f,t) := \left\{ \int_0^\infty \left| \nu^\alpha \{ s_\nu(f,t) - f(t) \} \right|^p \frac{d\nu}{\nu} \right\}^{1/p} \in L^\lambda \qquad (2.3)$$

if and only if

$$I(f,t) := \left\{ \int_0^\infty \left| \frac{f(t+u) - f(t-u)}{u^\alpha} \right|^p \frac{du}{u} \right\}^{1/p} \in L^\lambda. \qquad (2.4)$$

Assume $f \in L^r$ for some $1 \le r < \infty$ and $0 < p < \infty$. We define the strong approximation of order p of f by its Dirichlet integral as

$$\mathscr{S}_T^p(f,t) := \left\{ \frac{1}{T} \int_0^T |s_\nu(f,t) - f(t)|^p \, d\nu \right\}^{1/p}, \qquad T > 0, t \in \mathbf{R}.$$

By Hölder's inequality,

$$\mathscr{S}_T^p(f,t) \le \mathscr{S}_T^{p_1}(f,t) \qquad \text{if} \quad 0 < p < p_1 < \infty. \qquad (2.5)$$

It is easy to find the saturation order in L^λ-norm, where $0 < \lambda < \infty$. Indeed, if

$$\left\{ \int_{-\infty}^\infty [\mathscr{S}_T^p(f,t)]^\lambda \, dt \right\}^{1/\lambda} = o(T^{-1/p}) \qquad \text{as} \quad T \to \infty,$$

then we necessarily have

$$\int_0^\infty |s_\nu(f,t) - f(t)|^p \, d\nu = 0$$

for almost all $t \in \mathbf{R}$. Fix such t, then $s_\nu(f,t) = f(t)$ for all $\nu > 0$. Consequently, $f(t) = 0$ for almost all $t \in \mathbf{R}$. In other words, the saturation order is $\mathscr{O}(T^{-1/p})$.

Now, Theorem 2 makes it possible to determine the saturation class in L^λ-norm, where $2 \le p < \infty$ and $1 < \lambda < \infty$. Namely, if

$$\left\{\int_{-\infty}^{\infty} [\mathscr{S}_T^p(f,t)]^\lambda \, dt\right\}^{1/\lambda} = \mathscr{O}(T^{-1/p}) \qquad \text{as } T \to \infty,$$

then we necessarily have

$$\int_{-\infty}^{\infty} \left\{\int_0^{\infty} |s_\nu(f,t) - f(t)|^p \, d\nu\right\}^{\lambda/p} dt < \infty. \tag{2.6}$$

This coincides with (2.3) in the particular case where $\alpha = 1/p$. By virtue of Theorem 2, relation (2.6) is equivalent to (2.4) for $\alpha = 1/p$, $2 \le p < \infty$, $1 < \lambda < \infty$, which in turn is equivalent to the fact that $f \in F^\alpha_{\lambda,p}$, the so-called Lizorkin–Triebel space (see the remark made after Lemma 5 below).

Remark 2. The symmetric counterpart of Theorem 2 says that

$$\gamma_1(\tilde{f},t) = \left\{\int_0^{\infty} \left|\nu^\alpha\{\tilde{s}_\nu(f,t) - \tilde{f}(t)\}\right|^p \frac{d\nu}{\nu}\right\}^{1/p} \in L^\lambda \tag{2.7}$$

(cf. (1.17)) if and only if

$$I(\tilde{f},t) = \left\{\int_0^{\infty} \left|\frac{\tilde{f}(t+u) - \tilde{f}(t-u)}{u^\alpha}\right|^p \frac{du}{u}\right\}^{1/p} \in L^\lambda. \tag{2.8}$$

By virtue of Lemma 6, conditions (2.4) and (2.8) are equivalent. Hence, conditions (2.3) and (2.7) are also equivalent. In particular, this solves the problem of the strong approximation in L^λ-norm of the Hilbert transform $\tilde{f}$ by the conjugate Dirichlet integral, too.

3. Auxiliary Results

Let $z := x + iy$ with $y > 0$. By (1.1), (1.2), and (1.13),

$$\begin{aligned}\Phi(f,z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_0^{\infty} e^{i(z-t)u}\,du \\ &= \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt \\ &= U(f,x,y) + i\tilde{U}(f,x,y),\end{aligned} \tag{3.1}$$

where

$$U(f, x, y) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,dt$$

is the Poisson integral and

$$\tilde{U}(f, x, y) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{x-t}{(x-t)^2+y^2}\,dt$$

is the conjugate Poisson integral of f. Similarly to (1.17) and (2.1), we have

$$\tilde{U}(f, x, y) = U(\tilde{f}, x, y). \tag{3.2}$$

LEMMA 1. *Let $f \in L^r$ for some $1 \le r < \infty$, and let $z := x + iy$ with $y > 0$. Then*

$$i\Phi'(f, z) = \frac{\partial}{\partial y}U(f, x, y) + i\frac{\partial}{\partial y}\tilde{U}(f, x, y),$$

$$i^2\Phi''(f, z) = \frac{\partial^2}{\partial y^2}U(f, x, y) + i\frac{\partial^2}{\partial y^2}\tilde{U}(f, x, y).$$

Proof. Both equalities can be obtained by direct calculation starting with (3.1) and taking into account that

$$\Phi'(f, z) = \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-z)^2}\,dt,$$

$$\Phi''(f, z) = \frac{2}{i\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-z)^3}\,dt.$$

LEMMA 2 (see, e.g., [3, p. 569]). *Let $f \in L^p$ for some $1 < p < \infty$, and let $0 \le b < 1$, $0 \le a := b + p - 2$. If $|f(t)|^p|t|^a \in L^1$, then $\hat{f}_c$ exists in the sense of an ordinary function and*

$$\int_{-\infty}^{\infty}|\hat{f}_c(u)|^p|u|^{-b}\,du \le C_{b,p}\int_{-\infty}^{\infty}|f(t)|^p|t|^a\,dt.$$

An analogous conclusion holds if $\hat{f}_c$ is replaced by $\hat{f}_s$.

This inequality is known as Pitt's inequality. The special case where $a := 0$, $b := 2 - p$, and $1 < p \le 2$ is known as the Hardy–Littlewood inequality.

The next lemma is an extension of the famous inequality of M. Riesz from a single function to a sequence of functions.

LEMMA 3 (see, e.g., [3, p. 491]). *Let* $\{f_j(t)\colon j = 0, 1, \dots\}$ *be a sequence of functions belonging to* L^λ *for some* $1 < \lambda < \infty$ *and let* $1 < p < \infty$. *Then*

$$\left\| \left(\sum_{j=0}^{\infty} |\tilde{f}_j(t)|^p \right)^{1/p} \right\|_\lambda \le C_{p,\lambda} \left\| \left(\sum_{j=0}^{\infty} |f_j(t)|^p \right)^{1/p} \right\|_\lambda . \tag{3.3}$$

Here and in the sequel, we adopt the notation

$$\|f\|_\lambda := \left\{ \int_{-\infty}^{\infty} |f(t)|^\lambda \, dt \right\}^{1/\lambda} .$$

We note that the particular case where $p = 2$ and $1 < \lambda < \infty$ is an immediate consequence of a theorem of Marcinkiewicz and Zygmund (see, e.g., [3, p. 484]). Their reasoning can be slightly improved by interpolating with the obvious case: $1 < \lambda = p < \infty$. The general case stated in Lemma 3 is due to Boas and Bochner [1].

The next lemma is an extension of Lemma 3 from the discrete case to the continuous one.

LEMMA 4. *Let* $\{f_\nu(t)\colon \nu > 0\}$ *be a collection of functions belonging to* L^λ *for some* $1 < \lambda < \infty$ *and let* $1 < p < \infty$. *Furthermore, assume that, for almost all* $t \in \mathbf{R}$, *both* $f_\nu(t)$ *and* $\tilde{f}_\nu(t)$ *are continuous functions with respect to* ν *on the interval* $(0, \infty)$. *Then*

$$\left\| \left\{ \int_0^{\infty} |\tilde{f}_\nu(t)|^p \, d\nu \right\}^{1/p} \right\|_\lambda \le C_{p,\lambda} \left\| \left\{ \int_0^{\infty} |f_\nu(t)|^p \, d\nu \right\}^{1/p} \right\|_\lambda . \tag{3.4}$$

We conjecture that inequality (3.4) holds without the assumption of the continuity of $f_\nu(t)$ and $\tilde{f}_\nu(t)$ in ν. However, we will apply Lemma 4 in the case when $f_\nu(t) := \nu^{\alpha-(1/p)}\{s_\nu(f, t) - \sigma_\nu(f, t)\}$ (see (1.3) and (1.4)). Clearly, $f_\nu(t)$ is continuous, even analytic in ν for all $t \in \mathbf{R}$. The same is true for $\tilde{f}_\nu(t)$ (see (1.5), (1.6), and (1.17)), provided $1 < p < \infty$, which is the case in our Theorem 2.

Proof. Clearly, it is enough to deal with the case where

$$K := \left\| \left\{ \int_0^{\infty} |f_\nu(t)|^p \, d\nu \right\}^{1/p} \right\|_\lambda < \infty . \tag{3.5}$$

Then for almost all $t \in \mathbf{R}$ we have

$$\int_0^{\infty} |f_\nu(t)|^p \, d\nu < \infty.$$

Fix such $t \in \mathbf{R}$ with the additional property that both $f_\nu(t)$ and $\tilde{f}_\nu(t)$ are continuous with respect to ν on $(0, \infty)$.

Choose $0 < \varepsilon < A < \infty$ and consider a sequence of partitions

$$\varepsilon := t_0^k < t_1^k < \cdots < t_k^k =: A, \qquad k = 1, 2, \dots,$$

such that

$$\max_{1 \le j \le k} \left(t_j^k - t_{j-1}^k \right) \to 0 \qquad \text{as } k \to \infty.$$

Then the expressions involving Riemann sums

$$F_k(t) := \left\{ \sum_{j=1}^{k} \left| f_{\nu_j^k}(t) \right|^p \left(t_j^k - t_{j-1}^k \right) \right\}^{1/p}$$

and

$$G_k(t) := \left\{ \sum_{j=1}^{k} \left| \tilde{f}_{\nu_j^k}(t) \right|^p \left(t_j^k - t_{j-1}^k \right) \right\}^{1/p},$$

where $t_{j-1}^k \le \nu_j^k \le t_j^k$ for all j and k, converge to the corresponding Riemann integrals

$$F(t) := \left\{ \int_\varepsilon^A |f_\nu(t)|^p \, d\nu \right\}^{1/p} \qquad \text{and} \qquad G(t) := \left\{ \int_\varepsilon^A |\tilde{f}_\nu(t)|^p \, d\nu \right\}^{1/p},$$

respectively, as $k \to \infty$. This is true for almost all $t \in \mathbf{R}$.

By Lemma 3 and (3.5),

$$\| G_k(t) \|_\lambda \le C_{p,\lambda} \| F_k(t) \|_\lambda \le C_{p,\lambda} K.$$

By Fatou's lemma,

$$\| G(t) \|_\lambda := \left\| \left\{ \int_\varepsilon^A |\tilde{f}_\nu(t)|^p \, d\nu \right\}^{1/p} \right\|_\lambda \le C_{p,\lambda} K.$$

Letting $\varepsilon \to 0$ and $A \to \infty$, hence we conclude that

$$\left\| \left\{ \int_0^\infty |\tilde{f}_\nu(t)|^p \, d\nu \right\}^{1/p} \right\|_\lambda \le C_{p,\lambda} K. \tag{3.6}$$

Now, (3.4) follows from (3.5) and (3.6).

The equivalence statements formulated in Lemmas 5 and 7 below are due to Triebel [7]. Actually, we need their particular form corresponding to the case where

$$\varphi(t) := e^{it} - e^{-it}, \qquad \varphi_j(t) := \varphi(2^{-j}t), \qquad j = 0, 1, \dots,$$

and accordingly,

$$\varphi(uD)f(t) := f(t+u) - f(t-u).$$

(These notations are taken from [7].)

LEMMA 5 (see [7, pp. 100–101]). *Let $f \in L^\lambda$ for some $0 < \lambda < \infty$ and let $0 < p < \infty$, $\alpha \in \mathbf{R}$. Then* (2.4) *is satisfied if and only if*

$$\left\| \left(\sum_{j=0}^\infty \left| 2^{\alpha j} \{ f(t + 2^{-j}) - f(t - 2^{-j}) \} \right|^p \right)^{1/p} \right\|_\lambda < \infty. \tag{3.7}$$

We note that the Lizorkin–Triebel space $F^\alpha_{\lambda,p}$ is defined by (3.7). By Lemma 5, an equivalent norm in $F^\alpha_{\lambda,p}$ is defined by (2.4).

Let

$$I_*(f,t) := \left\{ \int_0^\infty \left| \frac{\Phi(f,t+u) - \Phi(f,t-u)}{u^\alpha} \right|^p \frac{du}{u} \right\}^{1/p}, \qquad t \in \mathbf{R}.$$

Combining the obvious inequalities

$$\max\{ I(f,t), I(\tilde{f},t) \} \le I_*(f,t) \le I(f,t) + I(\tilde{f},t)$$

with Lemmas 3 and 5 yields

LEMMA 6. *Let $f \in L^\lambda$ for some $1 < \lambda < \infty$ and let $1 < p < \infty$, $\alpha \in \mathbf{R}$. Then*

$$I(f) \in L^\lambda \Leftrightarrow I(\tilde{f}) \in L^\lambda \Leftrightarrow I_*(f) \in L^\lambda.$$

Following the pattern of $g_{2*}(f)$, let

$$g_2(f,t) := \left\{\int_0^\infty \left| y^{2-\alpha} \frac{\partial^2}{\partial y^2} U(f,x,y)\right|^p \frac{dy}{y}\right\}^{1/p}, \qquad t \in \mathbf{R}.$$

Then

$$g_2(\tilde{f},t) := \left\{\int_0^\infty \left| y^{2-\alpha} \frac{\partial^2}{\partial y^2} \tilde{U}(f,x,y)\right|^p \frac{dy}{y}\right\}^{1/p}$$

(cf. (3.2)).

LEMMA 7 (see [7, pp. 151–152]). *Let $f \in L^\lambda$ for some $0 < \lambda < \infty$ and let $0 < p < \infty$, $-\infty < \alpha < 2 - (1/\min(\lambda, p))$. Then*

$$I(f) \in L^\lambda \Leftrightarrow g_2(f) \in L^\lambda.$$

Combining the obvious inequalities

$$\max\{g_2(f,t), g_2(\tilde{f},t)\} \le g_{2*}(f,t) \le g_2(f,t) + g_2(\tilde{f},t)$$

with Lemmas 6 and 7 yields

LEMMA 8. *Let $f \in L^\lambda$ for some $1 < \lambda < \infty$ and let $1 < p < \infty$, $-\infty < \alpha < 2 - (1/\min(\lambda, p))$. Then*

$$g_2(f) \in L^\lambda \Leftrightarrow g_2(\tilde{f}) \in L^\lambda \Leftrightarrow g_{2*}(f) \in L^\lambda.$$

Following the pattern of $\gamma_*(f)$, let

$$\gamma(f,t) := \left\{\int_0^\infty |\nu^\alpha\{s_\nu(f,t) - \sigma_\nu(f,t)\}|^p \frac{d\nu}{\nu}\right\}^{1/p}, \qquad t \in \mathbf{R}.$$

Then

$$\gamma(\tilde{f},t) = \left\{\int_0^\infty |\nu^\alpha\{\tilde{s}_\nu(f,t) - \tilde{\sigma}_\nu(f,t)\}|^p \frac{d\nu}{\nu}\right\}^{1/p}$$

(cf. (2.1)). As a consequence of Lemma 4, we obtain the following

LEMMA 9. *Let $f \in L^\lambda$ for some $1 < \lambda < \infty$ and let $1 < p < \infty$, $\alpha \in \mathbf{R}$. Then*

$$\gamma(f) \in L^\lambda \Leftrightarrow \gamma(\tilde{f}) \in L^\lambda \Leftrightarrow \gamma_*(f) \in L^\lambda.$$

So far, we have defined the functions $\gamma(f)$, $\gamma_*(f)$, $\gamma_1(f)$, $g_2(f)$, and $g_{2*}(f)$. In Sections 4 and 5, we will define $\gamma_2(f)$, $g_{1*}(f)$, and $g_1(f)$. All these are called Littlewood–Paley functions (cf. [8, Vol. 2, Chaps. 14, 15]).

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. *Part* 1. By (1.3)–(1.6),

$$\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t) = \frac{1}{\nu}\int_0^\nu u\{\hat{f}_c(u) - i\hat{f}_s(u)\}e^{itu}\,du,$$

whence

$$\begin{aligned}\int_0^\infty &\nu\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}e^{iz\nu}\,d\nu\\ &= \frac{1}{z}\int_0^\infty iue^{i(t+z)u}\{\hat{f}_c(u) - i\hat{f}_s(u)\}\,du\\ &= \frac{\Phi'(f,t+z)}{z}, \qquad z := x+iy,\ y>0,\ t\in\mathbf{R}. \qquad (4.1)\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}\left|\frac{\Phi'(f,t+x+iy)}{x+iy}\right|^p &\le \int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p e^{-\nu y}\,d\nu\\ &\quad\times\left\{\int_0^\infty \nu^{(1-\alpha)q}e^{-\nu y}\,d\nu\right\}^{p-1}\\ &= \frac{C_{\alpha,p}}{y^{2p-1-\alpha p}}\int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p e^{-\nu y}\,d\nu,\end{aligned}$$

where

$$C_{\alpha,p} := (\Gamma((1-\alpha)q+1))^{p-1}, \qquad \frac{1}{p}+\frac{1}{q} = 1,$$

and Γ is the common gamma function. Hence

$$\begin{aligned}\int_0^\infty &y^{2p-1-\alpha p}\left|\frac{\Phi'(f,t+x+iy)}{x+iy}\right|^p dy\\ &\le C_{\alpha,p}\int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p d\nu\int_0^\infty e^{-\nu y}\,dy\\ &= C_{\alpha,p}\gamma_*^p(f,t), \qquad t\in\mathbf{R}.\end{aligned}$$

Setting $x := 0$ yields

$$g_{1*}(f,t) := \left\{\int_0^\infty \left| y^{1-\alpha}\Phi'(f,t+iy)\right|^p \frac{dy}{y}\right\}^{1/p}$$
$$\leq C_{\alpha,p}^{1/p}\gamma_*(f,t), \qquad t \in \mathbf{R}. \tag{4.2}$$

Part 2. Taking the derivatives of both sides in (4.1) with respect to z, we get

$$i\int_0^\infty \nu^2\{\tau_\nu(f,x) + i\tilde{\tau}_\nu(f,x)\}e^{iz\nu}\,d\nu$$
$$= \frac{\Phi''(f,t+z)}{z} - \frac{\Phi'(f,t+z)}{z^2}, \qquad z := x+iy,\ y>0,\ t\in\mathbf{R}. \tag{4.3}$$

By Hölder's inequality,

$$\left|\frac{\Phi''(f,t+z)}{z} - \frac{\Phi'(f,t+z)}{z^2}\right|^p$$
$$\leq \left\{\int_0^\infty \nu^2|\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)|e^{-\nu y}\,d\nu\right\}^p$$
$$\leq \int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p e^{-\nu y}\,d\nu \left\{\int_0^\infty \nu^{(2-\alpha)q}e^{-\nu y}\,d\nu\right\}^{p-1}$$
$$= \frac{C_{\alpha,p}}{y^{3p-1-\alpha p}}\int_0^\infty \left|\nu^\alpha\{\tau_\nu(f,t) + i\tilde{\tau}_\nu(f,t)\}\right|^p e^{-\nu y}\,d\nu,$$

where this time

$$C_{\alpha,p} := \left(\Gamma((2-\alpha)q+1)\right)^{p-1}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\int_0^\infty y^{3p-1-\alpha p}\left|\frac{\Phi''(f,t+z)}{z} - \frac{\Phi'(f,t+z)}{z^2}\right|^p dy \leq C_{\alpha,p}\gamma_*^p(f,t).$$

Setting $x := 0$ and making use of Minkowski's inequality yields

$$g_{2*}(f,t) \leq C_{\alpha,p}^{1/p}\gamma_*(f,t) + g_{1*}(f,t), \qquad t\in\mathbf{R}.$$

Taking into account (4.2), we conclude (2.2).

Proof of Theorem 2. *Necessity.* Assume (2.3). By Minkowski's inequality,

$$\gamma(f,t) \le \gamma_1(f,t) + \gamma_2(f,t), \tag{4.4}$$

where

$$\gamma_2(f,t) := \left\{\int_0^\infty |\nu^\alpha\{\sigma_\nu(f,t) - f(t)\}|^p \frac{d\nu}{\nu}\right\}^{1/p}, \qquad t \in \mathbf{R}.$$

By (1.11) and [4, Lemma 5], we get

$$\begin{aligned}\gamma_2(f,t) &= \left\{\int_0^\infty \left|\nu^{\alpha-1}\int_0^\nu \{s_\mu(f,t) - f(t)\}\, d\mu\right|^p \frac{d\nu}{\nu}\right\}^{1/p}\\ &\le (1-\alpha)^{-1}\gamma_1(f,t).\end{aligned} \tag{4.5}$$

Combining (2.3), (4.4), and (4.5) yields $\gamma(f) \in L^\lambda$. By Lemma 9, we have $\gamma_*(f) \in L^\lambda$. Applying Theorem 1 and Lemmas 8 and 7, we conclude (2.4).

Sufficiency. Assume (2.4). By (1.8) and (1.14), we have

$$\tilde{s}_\nu(f,t) - \tilde{f}(t) = \frac{1}{\pi}\int_{\to 0}^\infty \frac{f(t+u) - f(t-u)}{u} \cos \nu u\, du,$$

provided $\tilde{f}(t)$ exists, where $\int_{\to 0}^\infty := \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty$. Applying Lemma 2 with $b := 1 - \alpha p \ge 0$ and $a := p(1-\alpha) - 1 \ge 0$, we obtain

$$\begin{aligned}\gamma_1(\tilde{f},t) &= \frac{1}{\pi}\left\{\int_0^\infty \nu^{\alpha p - 1}\left|\int_{\to 0}^\infty \frac{f(t+u) - f(t-u)}{u} \cos \nu u\, du\right|^p d\nu\right\}^{1/p}\\ &\le C_{\alpha,p}^{1/p}\left\{\int_0^\infty \left|\frac{f(t+u) - f(t-u)}{u}\right|^p u^{p(1-\alpha)-1}\, du\right\}^{1/p}\\ &= C_{\alpha,p}^{1/p} I(f,t)\end{aligned} \tag{4.6}$$

for all $t \in \mathbf{R}$. The symmetric counterpart of (4.6) says that we have

$$\gamma_1(f,t) \le C_{\alpha,p}^{1/p} I(\tilde{f},t) \tag{4.7}$$

also for all $t \in \mathbf{R}$. It remains to apply Lemma 6 in order to conclude (2.3).

5. CONCLUDING REMARKS

Remark 3. As a by-product, we obtain that in Theorem 1 we actually have the equivalence relation

$$\gamma_*(f) \in L^\lambda \Leftrightarrow g_{2*}(f) \in L^\lambda,$$

provided $1 < p < \infty$, $1 < \lambda < \infty$, and $0 < \alpha \le \min(1/p, 1/q)$.

Indeed, it is enough to check the implication $\Leftarrow$. Assume $g_{2*}(f) \in L^\lambda$. By Lemmas 8 and 7, $I(f) \in L^\lambda$. By Theorem 2, $\gamma_1(f) \in L^\lambda$. By (4.4) and (4.5) in the proof of the Necessity in Theorem 2, we have $\gamma(f) \in L^\lambda$. Finally, by Lemma 9, we find the desired relation $\gamma_*(f) \in L^\lambda$.

Remark 4. Analogously, one can establish the equivalence relation

$$\gamma_*(f) \in L^\lambda \Leftrightarrow g_{1*}(f) \in L^\lambda$$

for specified values of p, λ, and α. The corresponding reasoning should involve the function

$$g_1(f,t) := \left\{ \int_0^\infty \left| y^{1-\alpha} \frac{\partial}{\partial y} U(f,x,y) \right|^p \frac{dy}{y} \right\}^{1/p}, \qquad t \in \mathbf{R}.$$

Remark 5. The machinery elaborated in this paper is also appropriate to characterize the saturation class concerning the strong approximation of order p of a periodic function $f \in L^1_{2\pi} := L^1(-\pi, \pi)$ by the partial sums $s_n(f,t)$ of its Fourier series in $L^\lambda_{2\pi}$-norm, where $2 \le p < \infty$ and $1 < \lambda < \infty$. A slight difference arises by the fact that $L^1_{2\pi} \supset L^r_{2\pi}$ for all $r > 1$, while $L^1(\mathbf{R})$ and $L^r(\mathbf{R})$ are incomparable.

THEOREM 3. *Let* $f \in L^1_{2\pi}$, $1 < p < \infty$, $1 < \lambda < \infty$, *and* $0 < \alpha \le \min(1/p, 1/q)$, *where* $1/p + 1/q = 1$. *Then*

$$\int_{-\pi}^{\pi} \left(\sum_{n=1}^\infty n^{\alpha p - 1} |s_n(f,t) - f(t)|^p \right)^{\lambda/p} dt < \infty \tag{5.1}$$

if and only if

$$\int_{-\pi}^{\pi} \left\{ \int_0^\pi \left| \frac{f(t+u) - f(t-u)}{u^\alpha} \right|^p \frac{du}{u} \right\}^{\lambda/p} dt < \infty. \tag{5.2}$$

The nth strong approximation of order p of f by the partial sums $s_k(f,t)$ of its Fourier series is defined by

$$\mathscr{A}_n^p(f,t) := \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_k(f,t) - f(t)|^p \right\}^{1/p}.$$

According to Theorem 3, the saturation class in L^λ-norm is the (periodic) Lizorkin–Triebel space $F_{\lambda,p}^{\alpha}$ on the torus, where $\alpha = 1/p$, $2 \le p < \infty$, and $1 < \lambda < \infty$.

The equivalence relation (5.1) $\Leftrightarrow$ (5.2) was proved by Sunouchi [5] for $1 < p \le \lambda < \infty$ and $\alpha = 1/p$. His proof hinges upon a lemma which is implicitly included in [2] by Flett. Unfortunately, there are two errors in [2]: see (4.11) and Theorem 11 on p. 368 as well as Theorem 20 on p. 374 (cf. what is said on p. 378).

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